The Sunrise Equation

Consider a point P on the globe as drawn in Fig 1A. This globe is not tilted. The latitude of the point is β. The coordinate system has its origin at the center of the globe. The axis which are draw just remind us of the relative orientation of the axis.

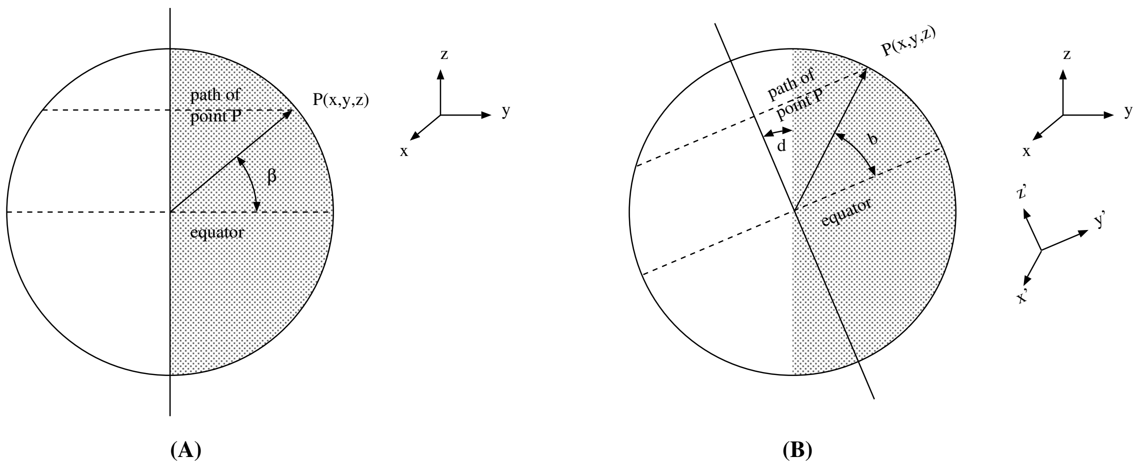

Figure 1: The path of point $P(x, y, z)$ on a globe which is (A) not tilted, (B) tilted.

Now we would like to write the equation for the location of point P in (x, y, z) as a function of time t. Note that the drawing show P at time $t = 0$, that is at midnight. So

$$
\begin{aligned}
P(x, y, z; t) \;=\; & (Rcos(\beta)sin(\omega t), \\
& Rcos(\beta)cos(\omega t), \\
& Rsin(\beta))
\end{aligned}
$$

As draw here, when $y > 0$ then it is night time for point P. If t is in hours, then $\omega t = 2\pi$ and $\omega = \pi/12$

Now we will rotate our coordinate system around the x-axis by angle δ as shown in Fig 1(B).

in general

$$p(x',y',z') = (x, ycos(\delta) - zsin(\delta), zcos(\delta) + ysin(\delta))$$

or for our equation for P

$$\begin{aligned} P(x',y',z';t) &= \Big(Rcos(\beta)sin(\omega t), \\ &\quad Rcos(\beta)cos(\omega t)cos(\delta) - Rsin(\beta)sin(\delta), \\ &\quad Rsin(\beta)cos(\delta) + (Rcos(\beta)cos(\omega t)sin(\delta)\Big) \end{aligned}$$

As draw in Fig 1B, when $y > 0$ then it is night time for point P. When $y = 0$ we have sunrise or sunset. Solve for that

$$Rcos(\beta)cos(\omega t)cos(\delta) - Rsin(\beta)sin(\delta) = 0$$

$$cos(\omega t) = \frac{sin(\beta)sin(\delta)}{cos(\beta)cos(\delta)} \tag{1}$$

or

$$cos(\omega t) = tan(\beta)tan(\delta) \tag{2}$$

$$t = \frac{1}{\omega}cos^{-1}\Big(tan(\beta)tan(\delta)\Big) \tag{3}$$

This is the "sunrise equation".

There are two missing effects. 1) The sun is not a point of light. It actually subtends 0.5^o. 2) The atmosphere curves the light slightly near the horizon. The effect of this is to mortify equation (3) to add an "altitude" correction.

$$cos(\omega t) = \frac{sin(\beta)sin(\delta) - sin(a)}{cos(\beta)cos(\delta)} \tag{4}$$

where $a = -0.83^o$

This is the "corrected" sunrise equation. This will give you the right sunrise/sunset time at the solstice.

The last step is the correct for day of year. This is done with the substitution

$$\alpha \longrightarrow \alpha cos(\omega_2 t_2)$$

where t_2 is days to the solstices and

$$\omega_2 = \frac{2\pi}{365\ days}$$

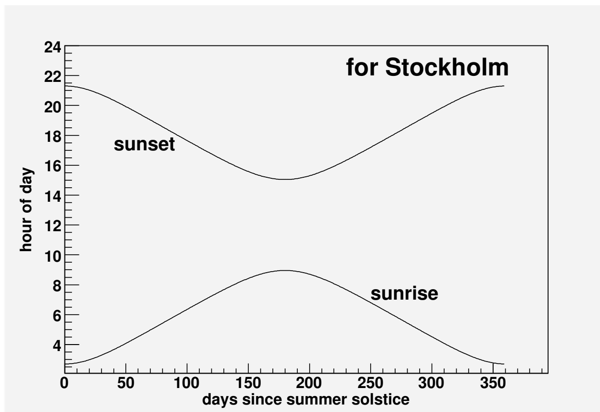

Figure 2: Sunrise and Sunset in Stockholm through the year.

Bright Skies

The last thing to calculate related to why is the sky still bright at midnight. Consider the last figure.

In the diagram above I know the radius of the Earth $R = 5370km$ and the latitudes of the arctic circle ($76^o33'$ or 76.55^o) and the latitudes of Stockholm ($59^o17'$ or 59.28^o), so the difference is $\alpha = 7.27^o$.

The angles β follow

$$\alpha + 2\beta = 90^o$$

$$\beta = \frac{180^o - \alpha}{2} = 86.37^o$$

$$\beta + \delta = 90^o \longrightarrow \delta = 90^o - \beta = 3.63^o$$

$$\beta + \gamma = 180^o \longrightarrow \gamma = 180^o - \beta = 93.62$$

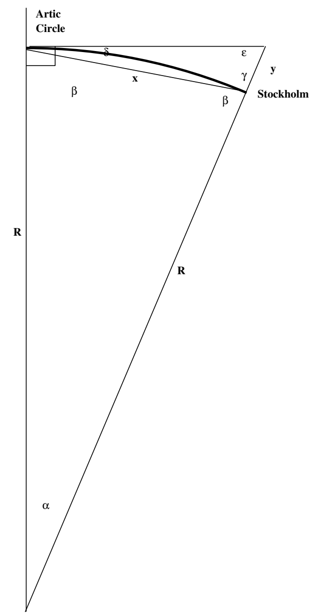

Figure 3: A slice through the center of the Earth, Stockholm and the Arctic Circle.

$$\delta + \gamma + \epsilon = 180^o \longrightarrow \epsilon = 180^o - \delta - \gamma = 82.73^o$$

Now I can solve for x with the law of sines

$$\frac{R}{sin(\beta)} = \frac{x}{sin(\alpha)}$$

$$\frac{Rsin(\alpha)}{sin(\beta)} = x = 807km$$

(note this is also close to $\frac{\alpha \times 10{,}000km}{90^o}$)
Finally, again using the law of sines

$$\frac{x}{sin(\epsilon)} = \frac{y}{sin(\delta)}$$

or

$$\frac{x sin(\delta)}{sin(\epsilon)} = y = 51.5km$$

I have ignored the fact that sunlight is curved by the atmosphere. Even so the sun is shining on the atmosphere a mere 51 km or 30 miles above me.